disc 105

Wednesday, March 3, 2021

3.5 Problems

Problem 1. *Determine the natural cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$.*

Problem 2. *Determine the clamped cubic spline s that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = s'(2) = 1$*

Problem 3. *Suppose $\{x_i f(x_i)\}_{i=1}^{n}$ lie on a straight line. What can be said about the natural and clamped cubic splines for the function f?*

Definition 3.10 Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a **cubic spline interpolant** S for f is a function that satisfies the following conditions:

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(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$;

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$;

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$; (*Implied by* **(b)**.)

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(f) One of the following sets of boundary conditions is satisfied:

(i) $S''(x_0) = S''(x_n) = 0$ (**natural** *(or free)* **boundary**);

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped boundary**).

1) find polynomials $P_1(x)$ on $[0,1]$ of degree ≤ 3

& $P_2(x)$ on $[1,2]$

s.t. $\cancel{P_1(0) = 0}$ $\quad P_1'(1) = P_2'(1)$

$P_1(1) = 1$ $\quad P_1''(1) = P_2''(1)$

$\cancel{P_2(1) = 1}$ $\quad P_1''(0) = 0$

$P_2(2) = 2$ $\quad P_2''(2) = 0$

$$P_1(x) = 0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$P_2(x) = 1 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3$$

$a_1 + a_2 + a_3 = 1$

$1 + b_1 + b_2 + b_3 = 2$

$2a_2 = 0$

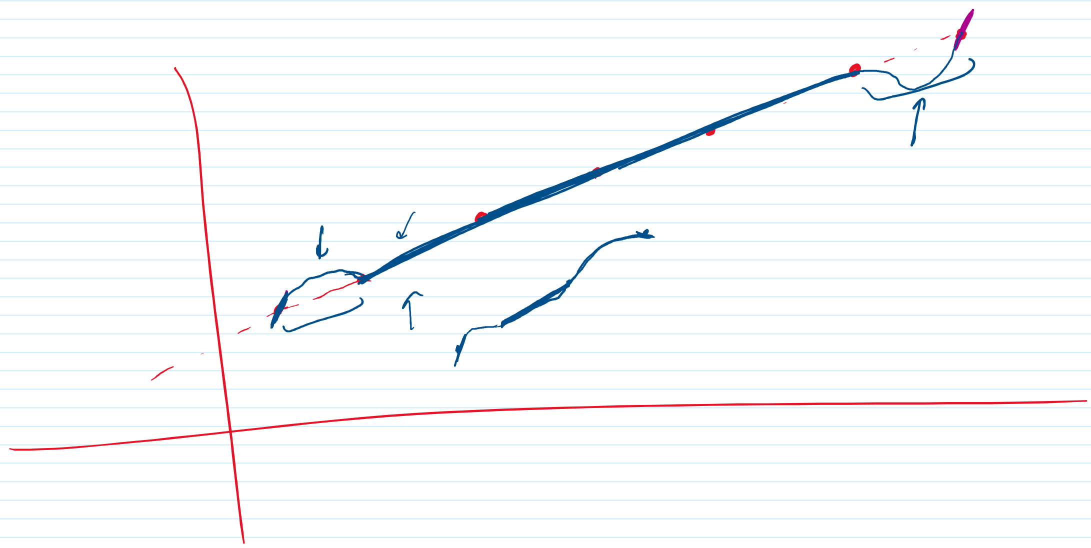
$2b_2 + 12 b_3 = 0$

$a_1 + 2a_2 + 3a_3 = b_1$

$2a_2 + 6a_3 = 2b_2$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 12 \\ 1 & 2 & 3 & -1 & 0 & 0 \\ 0 & 2 & 6 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Ax = b$$



Natural Splines

Theorem 3.11 If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$, then f has a unique natural spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$; that is, a spline interpolant that satisfies the natural boundary conditions $S''(a) = 0$ and $S''(b) = 0$.

Theorem 3.12 If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$; that is, a spline interpolant that satisfies the clamped boundary conditions $S'(a) = f'(a)$ and $S'(b) = f'(b)$.

3.6 Problems

Problem 4. *Let $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (5, 2)$ be the endpoints of a curve. Use the given guidepoints to construct parametric cubic Hermite approximations $(x(t), y(t))$ to the curve and graph the approximations (a) $(1, 1)$ and $(6, 1)$*

$X_0 = 0 \quad X_1 = 5$

$y_0 = 0 \quad y_1 = 2$

$(1,1) = (X_0 + \alpha_0, y_0 + \beta_0)$

$= (\alpha_0, \beta_0)$

$\Rightarrow \alpha_0 = 1$

In Figure 3.17, the nodes occur at (x_0, y_0) and (x_1, y_1), the guidepoint for (x_0, y_0) is $(x_0 + \alpha_0, y_0 + \beta_0)$, and the guidepoint for (x_1, y_1) is $(x_1 - \alpha_1, y_1 - \beta_1)$. The cubic Hermite polynomial $x(t)$ on $[0, 1]$ satisfies

$$x(0) = x_0, \quad x(1) = x_1, \quad x'(0) = \alpha_0, \quad \text{and} \quad x'(1) = \alpha_1.$$

The unique cubic polynomial satisfying these conditions is

$$x(t) = [2(x_0 - x_1) + (\alpha_0 + \alpha_1)]t^3 + [3(x_1 - x_0) - (\alpha_1 + 2\alpha_0)]t^2 + \alpha_0 t + x_0. \quad (3.23)$$

In a similar manner, the unique cubic polynomial satisfying

$$y(0) = y_0, \quad y(1) = y_1, \quad y'(0) = \beta_0, \quad \text{and} \quad y'(1) = \beta_1$$

is

$$y(t) = [2(y_0 - y_1) + (\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - (\beta_1 + 2\beta_0)]t^2 + \beta_0 t + y_0. \quad (3.24)$$

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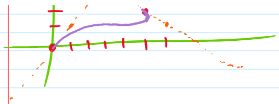

$(1,1) = (x_0+\alpha_0, y_0+\beta_0)$

$= (\alpha_0, \beta_0)$

$\Rightarrow \boxed{\alpha_0 = 1 \quad \beta_0 = 1}$

$(6,1) = (x_1-\alpha_1, y_1-\beta_1)$

$= (5-\alpha_1, 2-\beta_1)$

$\Rightarrow \boxed{\alpha_1 = -1 \quad \beta_1 = 1}$

is

$$y(t) = [2(y_0 - y_1) + (\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - (\beta_1 + 2\beta_0)]t^2 + \beta_0 t + y_0. \quad (3.24)$$

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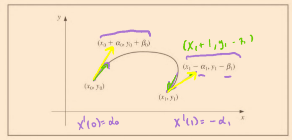
Figure 3.17



Find cubic polynomials $x(t)$ $y(t)$

s.t. $x(0)=0$, $x(1)=5$, $x'(0)=1$, $x'(1)=-1$

s.t. $y(0)=0$, $y(1)=2$, $y'(0)=1$, $y'(1)=1$

$\boxed{x(t) = (2(-5)+0)t^3 + (3\cdot 5 - 1)t^2 + t + 0 = -10t^3 + 14t^2 + t}$

$\boxed{y(t) = -2t^3 + 3t^2 + t}$

4.1 Problems

Problem 5. *Use the forward-difference formulas and backward-difference formulas to determine each missing entry in the following table:*

x	$f(x)$	$f'(x)$
.5	.4794	
.6	.5646	
.7	.6442	

Problem 6. *For the previous problem, $f(x) = \sin(x)$. Determine the actual error and find error bounds using the error formulas.*

Actual error $\cong .03$

$$\boxed{f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi).} \quad (4.1)$$

For small values of h, the difference quotient $[f(x_0+h) - f(x_0)]/h$ can be used to approximate $f'(x_0)$ with an error bounded by $M|h|/2$, where M is a bound on $|f''(x)|$ for x between x_0 and x_0+h. This formula is known as the **forward-difference formula** if $h > 0$ (see Figure 4.1) and the **backward-difference formula** if $h < 0$.

$$\left| \frac{f(x_0+h) - f(x_0)}{h} - f'(x_0) \right| = \frac{h}{2} |f''(\xi)|$$

$$= \frac{h}{2} |\sin(\xi)| \le \frac{h}{2} \cdot 1 = \frac{0.1}{2} = \frac{1}{10\cdot 2} = \frac{1}{20} = .05$$

$$\frac{\frac{1}{10}}{2} = \left(\frac{1}{10}\right)\left(\frac{1}{2}\right) = \frac{1}{10\cdot 2}$$